Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 4

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Textbook Sections: 6.1, 6.2, 6.3

Topics for Today: Computing Areas and Volumes using Integrals

The definition of the Riemann integral $\int_a^b f(x)dx$ in terms of a limit of Riemann sums suggests that we should be able to use integrals to compute areas/volumes of regions via integration. In Lectures 4 and 5, we'll explore this for a nice variety of regions.

I Areas Between Curves:

Recall that if $f:[a,b]\to\mathbb{R}$ is cts, then

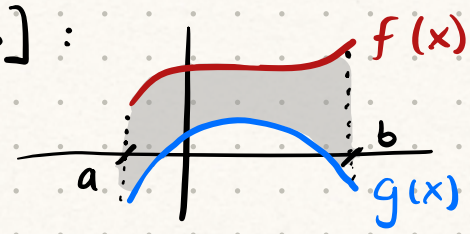
$\int_a^b f(x)dx$ = signed area under the graph of $f(x)$



In this set-up, we considered area "under the x-axis" as "negative area."

Let's consider a new sort of computation:

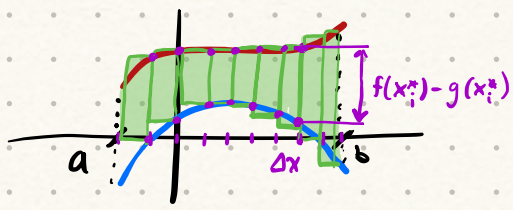
Imagine we want to find the area of a region, whose top/bottom boundaries are given by the graphs of two functions $f(x)\geq g(x)$, over the interval $[a,b]$:



If we approximate the area of this region using rectangles, we see that

$$\text{Area of region} \approx \sum_{i=1}^{N} (\text{height of rectangle } i) \cdot \Delta x$$

$$= \sum_{i=1}^{N} (f(x_i^*) - g(x_i^*)) \Delta x$$



and as we take $N \to \infty$, we see that

$$\text{Area of region} = \int_a^b [f(x) - g(x)]\, dx.$$

⚠ **Note:** In this application of integration, we *don't* count regions below the x-axis as having "negative area" like we did for "signed area". **Motto:** signed area can be negative, but just area cannot!

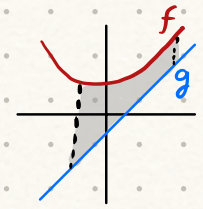
⚠ **About left and right boundaries:** In the above cartoon, the left/right-hand boundaries of the region were given by the lines $x = a$ and $x = b$. In other examples, this may not be the case!

⚠ **About the condition $f(x) \geq g(x)$:** This condition only means that the top of the region is given by the graph of $f(x)$. In general, we can have pieces of a region where $f(x) \geq g(x)$ and pieces where $g(x) \geq f(x)$ — we just have to adjust our integral set-up accordingly.

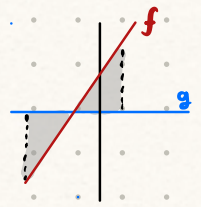
⚠ **More set-ups are possible!** The trick in doing such problems is to cleverly define the region using functions that you can then apply calculus to.

Lots of examples to illustrate all these points:

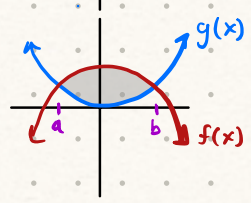
Find the area of the regions bounded by:



① $f(x) = x^2 + 4$ and $g(x) = x - 2$ over $[-1, 3]$



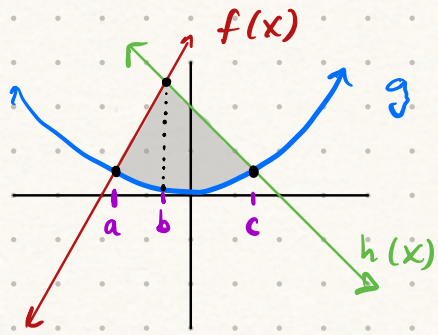
② $f(x) = 2x + 4$ and the x-axis over $[-5, 1]$



③ $f(x) = x^2$ and $g(x) = 8 - x^2$ (Note no endpoints are given!)

Note: When no endpoints are given, we always mean the bounded region!

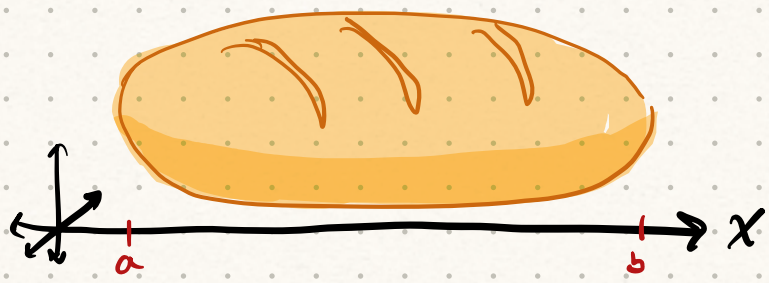
(4) $f(x) = 3x+4$, $g(x) = x^2$, and $h(x) = -x+2$ over $[-1,1]$.



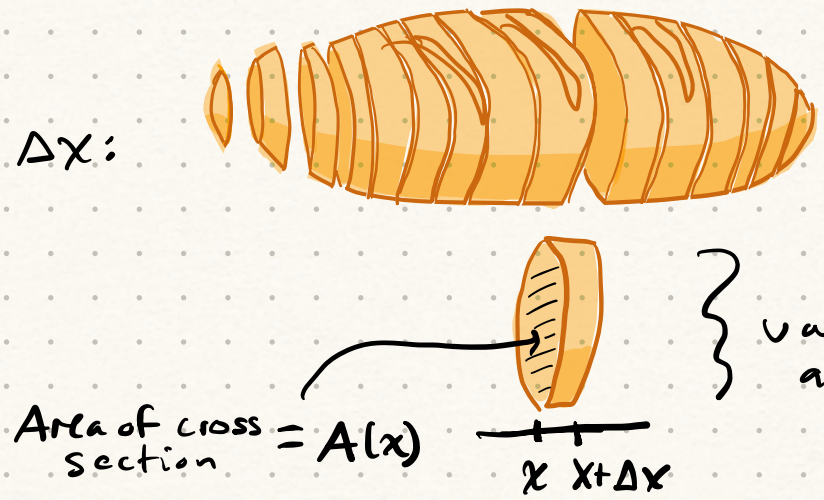
II Computing Volumes by Slicing

Here's another application of all these ideas (which generalizes I to three dimensions).

Suppose we want to find the volume of a loaf of bread:

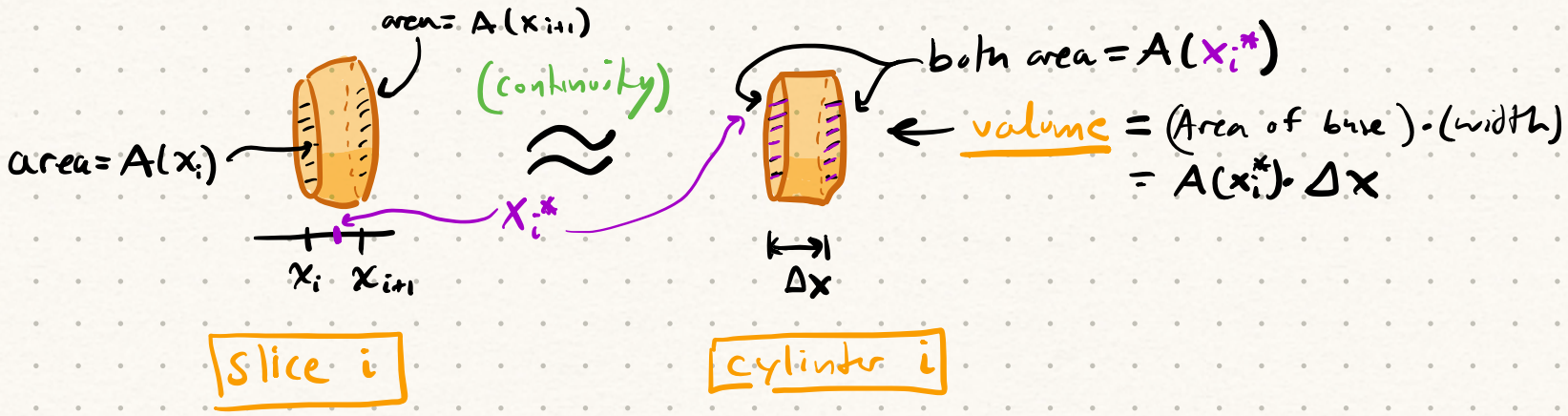


To approximate the volume, we could try slicing the bread into slices of width Δx:



So, if we approximate the volume of each (thin) slice, the sum of all these volumes will be close to the volume of the loaf.

Key Point: If the slices are very thin, then their individual volumes are easy to approximate, since they are close to being cylinders (the cross-section of the slice ought to change continuously along the loaf):



Thus, $(\text{volume of loaf}) = \sum_{i=1}^{N} (\text{volume of slice } i)$

$$\approx \sum_{i=1}^{N} (\text{volume of cylinder } i)$$

$$= \sum_{i=1}^{N} A(x_i) \cdot \Delta x$$

$$= \sum_{i=1}^{N} A(x_i) \frac{b-a}{N}$$

As we send $N \to \infty$, the approximation gets better and better, and we recognize this as a Riemann Sum with the function $A(x)$! Taking the limit, we thus obtain:

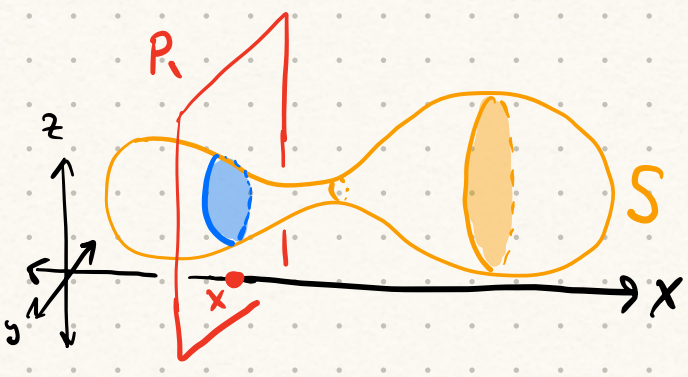
Definition/Theorem:

- Let S be a solid that lies between $x=a$ and $x=b$.
- Let $A(x)$ = area of the slice of S by the plane P_x perpendicular to the x-axis at x

If $A(x)$ is continuous (or at least "nice enough"),

$$(\text{volume of } S) := \lim_{N \to \infty} \sum_{i=1}^{N} A(x_i^*) \Delta x$$

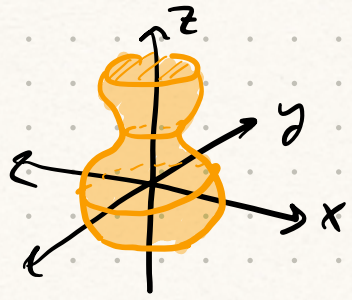
$$= \int_a^b A(x)\, dx .$$



General Remarks: (examples of each of these to follow)

(1) This formula is easy to apply once we can find a formula for the cross-sectional area $A(x)$.

(2) There is nothing special about using the variable x to "slice up the region." We could also slice the region in other directions too – such as in the z-direction. This could make problems easier!

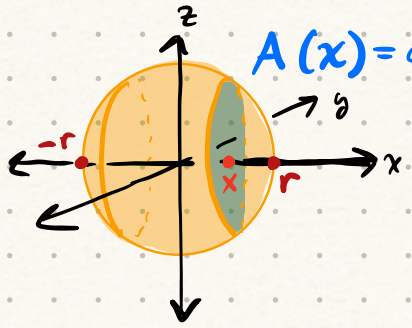


(3) Nothing in this definition requires the region S to have a symmetry.

Examples:

(1) Computing the Volume of the 3-dimensional ball of radius r.

Let $S =$



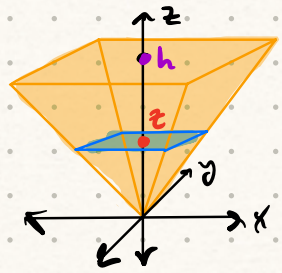
$A(x)$ = area of circle of radius $\sqrt{r^2 - x^2}$

$= \pi(\sqrt{r^2-x^2})^2$

$= \pi(r^2 - x^2)$

So volume of $S = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2-x^2)\,dx = \frac{4}{3}\pi r^3$

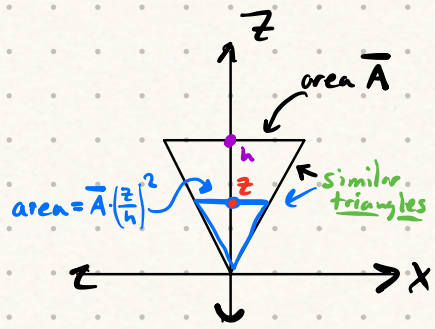
(2) Find the volume of a pyramid with a base of area $\overline{A}$ lying in the plane $z=h$, and with vertex at the origin:



Here: • we slice in the z-direction $(0 \le z \le h)$

• $A(z)$ = area of cross section at height z

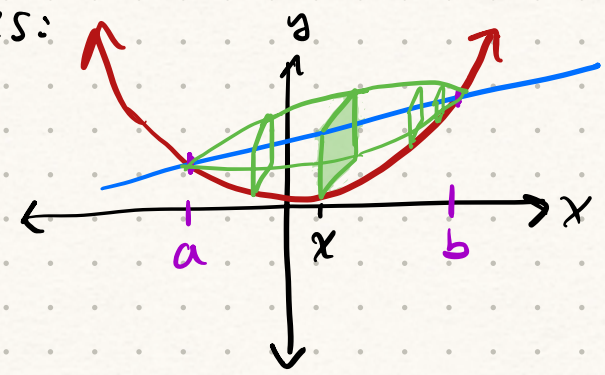
$= \overline{A} \cdot \left(\frac{z}{h}\right)^2$



So: $\text{Volume} = \int_0^h A(z)\,dz = \int_0^h \overline{A}\left(\frac{z}{h}\right)^2 dz = \frac{1}{3}\overline{A} \cdot h$

(notice how this generalizes the formula for the area of a triangle!)

(3) Let S be the solid whose base is the region bounded by $y = x^2$ and $y = x+2$, and whose cross-sectional slices in the x-direction are perfect squares:



Here:

• $a = -1$, $b = 2$, and $-1 \le x \le 2$.

• $A(x)$ = area of square of side length $(x+2) - x^2$

$= (x+2-x^2)^2$

So, $\text{volume} = \int_{-1}^{2} A(x)\,dx = \frac{81}{10}$ (units)^3.